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The Harmful Effects of Algorithms in Grades 1–4

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STARTING in the 1970s, researchers such as Ashlock (1972, 1976, 1982) and Brown and Burton (1978) have documented the erroneous but consistent ways in which children inadvertently change the algorithms for multidigit computation. The rules children made up showed that their focus was on trying to remember the steps instead of on logically solving the problems.

Although some researchers studied children's unsuccessful efforts to use the conventional algorithms, others reported the surprising procedures invented by many children, adolescents, and adults. (Throughout this paper, the term *algorithm* is used to refer to the conventional rules of "carrying," "borrowing," etc.; child-invented procedures are referred to as *procedures.*) Cochran, Barson, and Davis (1970), for example, described how an eight-year-old solved 62 – 28: first, 60 – 20 = 40, then, 2 – 8 = –6, and finally, 40 – 6 = 34. Similar findings have been reported in Argentina (Ferreiro 1988, personal communication 1976), the Netherlands (ter Heege 1978), England (Plunkett 1979), and South Africa (Murray and Olivier 1989).

By the 1980s, some researchers were seriously questioning the wisdom of teaching conventional algorithms. In Brazil, Carraher, Carraher, and Schliemann (1985) and Carraher and Schliemann (1985) compared children who used algorithms with those who used their own procedures. Posing as a customer, for example, the researcher asked a child street vendor how much four coconuts would cost if one cost 35 cruzeiros. The vendor replied, "Three will be 105, plus ... 35 ... 140!" (Carraher, Carraher, and Schliemann 1985, p. 26). In a subsequent interview, however, the same child wrote the answer 200, as shown in figure 17.1, and explained, "Four times 5 is 20, carry the 2; 2 plus 3 is 5, times 4 is 20" (p. 26). The researchers concluded that children who use their own procedures are much more likely to produce correct answers than those who try to use algorithms. They thus began to think that algorithms were a hindrance

$$\begin{array}{r} {\scriptstyle 2} \\ 35 \\ \times\ 4 \\ \hline 200 \end{array}$$

Fig. 17.1. A Brazilian child's way of using the algorithm

rather than a help. Jones (1975) in England, Vakali (1984) in Greece, and Dominick (1991) in the United States reached the same conclusion.

Some investigators went further in the 1990s and concluded that algorithms are harmful to children. Narode, Board, and Davenport (1993) compared second graders before and after they were taught algorithms and concluded that children lose conceptual knowledge when they learn these rules. Kamii (1994) compared children in grades 2–4 who had been taught algorithms with those who had never been taught any algorithms and found that those who did their own thinking got more correct answers and had much better knowledge of place value. She also pointed out that the algorithms that are now conventional are the results of centuries of construction by adult mathematicians. Although it is not necessary for children to go through every historical step, it is unrealistic to expect them to skip the entire process of construction.

Many algorithms that were conventional centuries ago reveal a parallel between an individual's construction of numerical reasoning and humanity's construction of these rules. For example, some Hindus added 278 and 356 on a "dust" board in the following way (Groza 1968):

278 -------	578 -------	628 -------	634
356	56	6	

In this algorithm, 200 and 300 of the 278 and 356 were added first and erased, and changed to 500 (the "5" of 578). The next step was to add 70 and 50, erase them and the 500, and change them to 620 (the "62" of 628). The 8 and 6 were then added and erased, as well as the 2, and changed to 34.

Some leaders in mathematics education also began to say that we must stop teaching algorithms because they make no sense to most children and discourage logical thinking. The most convincing arguments based on systematic study of children in classrooms were advanced by Madell (1985), Burns (1994), and Leinwand (1994).

The purpose of this paper is to present the evidence that led us to the conviction that algorithms not only are not helpful in learning arithmetic but also hinder children's development of numerical reasoning. We begin by discussing our data and go on to describe teachers' observations.

RESEARCH BASED ON PIAGET'S CONSTRUCTIVISM

The distinction Piaget made among the three kinds of knowledge on the basis of their ultimate sources shows why teaching conventional algorithms

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does not foster children's learning of mathematics. The three kinds of knowledge he distinguished are physical, social, and logico-mathematical knowledge.

Physical knowledge is knowledge of objects in external reality. The color and weight of a block are examples of physical properties that are *in* objects in external reality and can be known empirically by observation.

Examples of *social (conventional) knowledge* are holidays, such as the Fourth of July, and written and spoken languages, such as the word *block*. Whereas an ultimate source of physical knowledge is in objects, an ultimate source of social knowledge is in conventions made by people.

Logico-mathematical knowledge consists of mental relationships, and the ultimate source of these relationships is each person's mental actions. For example, the child's knowing that one quantity combined with another gives a larger quantity results from his or her making a mental relationship. Someone else can explain this relationship, but this explanation does not become the child's knowledge until he or she makes the relationship. Likewise, an adult can explain to a child the algorithm for two-digit addition. However, listening to this explanation does not ensure the child's making the mental relationships about how to combine the two quantities.

A characteristic of logico-mathematical knowledge is that there is nothing arbitrary in it. For example, adding 356 to 278 results in 634 in every culture. The social (conventional) rule, or algorithm, stating that one *must* add the ones first, then the tens, and then the hundreds is arbitrary. The teaching of algorithms is based on the erroneous assumption that mathematics is a cultural heritage that must be *transmitted* to the next generation.

Piaget's constructivism and the more than sixty years of scientific research by him and others all over the world led Kamii to a compelling hypothesis: Children in the primary grades should be able to invent their own arithmetic without the instruction they are now receiving from textbooks and workbooks. This hypothesis was amply verified, as can be seen in Kamii (1985, 1989b, 1994).

A significant byproduct of this research was the finding that when children are encouraged to do their own thinking to add, subtract, and multiply multidigit numbers, they always deal with the large units first, such as the tens, and then with the ones. As can be seen in figure 17.2 and Kamii (1989a, 1989b, 1994), this finding confirmed Madell's (1985) statement that when children are encouraged to think in their own ways, they "*universally* proceed from left to right" (p. 21).

Another significant finding was that at the end of second and third grade, the children in "constructivist" classes consistently excel over those in traditional classrooms where algorithms are taught. Hypothesizing that algorithms are harmful to children, Kamii compared the performance of children who had never been taught these conventional rules with those who had.

18 +17	10 + 10 = 20 8 + 7 = 15 20 + 10 = 30 30 + 5 = 35	10 + 10 = 20 8 + 2 = another ten 20 + 10 = 30 30 + 5 = 35	10 + 10 = 20 7 + 7 = 14 14 + 1 = 15 20 + 10 = 30 30 + 5 = 35
44 −15	40 − 10 = 30 4 − 5 = 1 less than 0 30 − 1 = 29	40 − 10 = 30 30 − 5 = 25 25 + 4 = 29	40 − 10 = 30 30 + 4 = 34 34 − 5 = 29
135 × 4	4 × 100 = 400 4 × 30 = 120 4 × 5 = 20 400 + 120 + 20 = 540	4 × 100 = 400 4 × 35 = 70 + 70 = 140 400 + 140 = 540	

Fig. 17.2. Procedures invented by children for addition, subtraction, and multiplication

In the school where she worked in 1989–91, some teachers taught algorithms whereas others did not, according to the following distribution:

Grade 1: None of the four teachers taught algorithms.

Grade 2: One of the three teachers taught algorithms; of the two who did not, one convinced parents that they should not teach algorithms at home, either.

Grade 3: Two of the three teachers taught algorithms.

Grade 4: All four teachers taught algorithms.

All the classes were heterogenous and comparable (the principal mixed up all the children at each grade level and divided them as randomly as possible before each school year). Students who transferred in from other schools were also distributed randomly among all the classes.

One of the problems Kamii asked each child to solve mentally in individual interviews was 7 + 52 + 186 (or 6 + 53 + 185). The answers given by the second, third, and fourth graders are presented in tables 17.1–17.3. It can be seen in tables 17.1 and 17.2 that the "No algorithms" classes, both in second and in third grade, produced the highest percentages of correct answers (45 percent and 50 percent respectively). It is also evident that the "No algorithms" second- and third-grade classes produced more correct answers than all the fourth-grade classes (table 17.3), who were all taught algorithms.

More significant are the incorrect answers listed in tables 17.1–17.3. All the wrong answers given in each class appear in these tables. The broken lines through the middle of each table indicate a range of answers that can be considered reasonable. It can be seen in table 17.1 that the incorrect answers of the "No algorithms" class were much more reasonable than the wrong answers of the "Algorithms" class. (The class that was exposed to some algorithms at home came out in between.) In third grade (table 17.2), too, the incorrect answers of the "No algorithms" class were much more reasonable than those of the "Algorithms" classes. The fourth graders (table 17.3), who had had an additional year of algorithms, gave incorrect answers that were more unreasonable than those of the third-grade "Algorithms" classes. In fourth grade, there were more answers in the 700s and 800s, and some four- and five-digit answers. A new symptom also emerged in fourth grade: Answers such as "4, 4, 4" consisting of single digits, demonstrating that children were thinking about three independent columns.

Table 17.1
Answers to 7 + 52 + 186 Given by Three Classes of Second Graders in May 1990

Algorithms $n = 17$	Some algorithms taught at home $n = 19$	No algorithms $n = 20$
9308		
1000		
989		
986		
938	989	
906	938	
838	810	
295	356	617
		255
		246
245 (12%)	245 (26%)	245 (45%)
		243
		236
		235
200	213	138
198	213	—
30	199	—
29	133	—
29	125	—
—	114	
—	—	
	—	
	—	

Note. Blanks indicate that the child declined to try to work the problem.

Table 17.2
Answers to 6 + 53 + 185 Given by Three Classes of Third Graders in May 1991

Algorithms $n = 19$	Algorithms $n = 20$	No algorithms $n = 10$
	800 + 38	
838	800	
768	444	
533	344	284
246		245
244 (32%)	244 (20%)	244 (50%)
235	243	243
234	239	238
	238	
	234	
213	204	221
194	202	
194	190	
74	187	
29	144	
—	139	
—	—	
	—	

Note. Blanks indicate that the child declined to try to work the problem.

Why Algorithms Are Harmful

We have two reasons for saying that algorithms are harmful: (1) They encourage children to give up their own thinking, and (2) they "unteach" place value, thereby preventing children from developing number sense.

As stated earlier, when children invent their own procedures, they proceed from left to right. Because there is no compromise possible between going toward the right and going toward the left as the algorithms require, children have to give up their own thinking to use the algorithms.

When we listen to children using the algorithm to do

$$\begin{array}{r} 89 \\ \underline{+34} \end{array},$$

for example, we can hear them say, "Nine and four is thirteen. Put down the three; carry the one. One and eight is nine, plus three is twelve...." The algorithm is convenient for adults, who already know that the "one," the "eight," and the "three" stand for 10, 80, and 30. However, for primary school children, who have a tendency to think that the "8" means eight, and so on, the algorithm serves to reinforce this error. The incorrect answers given by the "Algorithms" classes in tables 17.1–17.3 demonstrate that the algorithms

Table 17.3
Answers to 6 + 53 + 185 Given by Four Classes of Fourth Graders in May 1991

Algorithms $n = 20$	Algorithms $n = 21$	Algorithms $n = 21$	Algorithms $n = 18$
	1215		
	848		
	844		
	783		
1300	783		10,099
814	783		838
744	718	791	835
715	713	738	745
713 + 8	445	721	274
	245		
244 (30%)	244 (24%)	244 (19%)	244 (17%)
243	234		234
	224		234
			234
194	194	144	225
177	127	138	"8, 3, 8"
144	—	134	"4, 3, 2"
143	—	"8, 3, 7"	"4, 3, 2"
134		"8, 1, 7"	—
"4, 4, 4"		—	—
"1, 3, 2"		—	—
		—	
		—	
		—	
		—	
		—	
		—	
		—	

Note. Blanks indicate that the child declined to try to work the problem.

"untaught" place value and prevented the children from developing number sense. The children in all the "Algorithms" classes did not notice that their answers of 144, 783, and so on, were unreasonable for 6 + 53 + 185.

Most of the children in the "No algorithms" classes typically began by saying, "One hundred eighty and fifty is two hundred thirty." This is why their errors were reasonable even when they got an incorrect answer. Those in the "Algorithms" classes, however, typically said, "Six and three is nine, plus five is fourteen. Put down the four; carry the one...." Many of them then added 6 (the first addend) to the 1 in 185 (the third addend) and got an answer in the 700s [illegible] 900s.

Observations in Classrooms

The harmful effects of algorithms became even more evident when, in 1991–92, one of the fourth-grade teachers, Cheryl Ingram, decided to change her teaching to a constructivist approach. One of the ways in which she tried to wean the children away from algorithms was to write problems such as 876 + 359 horizontally on the board and to ask the class to invent many different ways of solving them without using a pencil. As the children volunteered to explain how they got the answer of 1235 by using the algorithm in their heads, she wrote exactly what students said for each column (6 + 9 = <u>15</u>, 7 + 5 + 1 = <u>13</u>, and 8 + 3 + 1 = <u>12</u>) as follows:

$$\begin{array}{r} 15 \\ 13 \\ \underline{+12} \\ 40 \end{array}$$

After the child finished explaining how he or she got the answer of 1235, Ms. Ingram said, "But I followed your way, and when I put 15, 13, and 12 together, I got 40 as my answer. How did you get 1235?" Most of the children were stumped and became silent, until someone pointed out that the teacher's 13 was really 130 and that her 12 stood for 1200.

This kind of place-value problem was not too hard to correct. The persisting difficulty lay in the column-by-column, single-digit approach that prevented children from thinking about multidigit numbers. Presented with problems like 876 + 359, the children continued to give fragmented answers from right to left, such as "5, 130, 1200" (for 6 + 9, 10 + 70 + 50, and 100 + 800 + 300, respectively).

One day, to encourage the children to think about multidigit numbers, Ms. Ingram put on the board one problem after another that had 99 (or 98 or 95) in one of the addends, such as 366 + 199, 493 + 99, and 601 + 199. Only this kind of problem was presented during the entire hour, and the children were asked as usual to think about different ways of adding them.

Almost all the children in the class continued to use the algorithm during the entire hour. One child, however, whom we will call Joe, had been in "constructivist" classes since first grade and volunteered solutions like the following for each problem: "I changed '366 + 199' to '365 + 200,' and my answer is 565." After an entire hour of this kind of "interaction," the number of children imitating Joe by the end of the hour was only three! The rest of the class continued to deal with each column separately.

The school year proceeded with many ups and downs as Ms. Ingram continued her struggle to revive the children's own thinking (see Kamii [1994] for further detail). In May 1992, 6 + 53 + 185 was presented to her fourth graders, and the results were gratifying, as can be seen in figure 17.3.

In figure 17.3, the top row of each 2 × 2 matrix shows the number of children who gave the correct answer, and the bottom row shows t[illegible]umber of

1991

	Algorithm	Invented Procedures
Correct Answer	3	0
Incorrect Answers	13	1

1992

	Algorithm	Invented Procedures
Correct Answer	0	15
Incorrect Answers	2	3

(One child was excluded from this analysis because she said she was thinking of multiplying 185 by 53 and of adding 6.)

Fig. 17.3. Fourth graders' use of the algorithm and invented procedures and the correctness of their answer to 6 + 53 + 185 in May 1991 and May 1992

those who gave incorrect answers. The left-hand column of each matrix indicates the number who used the conventional algorithm, and the right-hand column shows the number who used their own invented procedures. By comparing these matrices, we can see that when Ms. Ingram taught algorithms in 1990–91, almost all her students used the algorithm, and most of them got incorrect answers (shown in the last column of table 17.3). In 1991–92, by contrast, when Ms. Ingram encouraged her students to do their own thinking, most of her students used invented procedures and got the correct answer.

Ann Dominick, the second author, is a classroom teacher who has taught third and fourth graders for twelve years. When she worked with fourth graders in one school, almost every child in her class had been taught algorithms before coming to her. Now that she teaches third graders in another school, most of the children in her class have *never* been taught these algorithms. The difference in students' thinking is astounding.

The most striking differences are in students' confidence and their knowledge of place value. Those who have made sense of mathematics approach it with confidence rather than fear and hesitation. The students' intellectual pace is a gallop instead of a walk.

At the beginning of each year Ms. Dominick conducts individual interviews with each student to assess, among other things, their knowledge of place value. In the place-value task (Kamii 1989b), children are asked to show with sixteen counters what each digit in the numeral 16 means. When students came to her class using algorithms, approximately 20 percent of the fourth graders each year showed ten counters for the 1 in 16. (The other 80 percent showed only one counter.) In the third-grade classes where most of the children have not been taught any algorithms, about 85 percent show ten counters for the 1 in 16.

Ms. Dominick's thinking about teaching algorithms has changed over the years from (*a*) teaching arithmetic *by* teaching algorithms to (*b*) teaching the algorithms after "laying the groundwork for understanding" to (*c*) not teaching algorithms at all. The final shift came from reflecting on what happened when she "laid the groundwork for understanding" and then taught the algorithm.

The first rationale for teaching the algorithms was that it seemed to be the most efficient method. Once students started inventing their own methods, however, this argument no longer held true. For example, using the algorithm for multiplying by 25 often *slows* students' thinking. Frequently, children use their knowledge of 25 × 4 = 100 to reason that 25 × 16 = 400. Similarly, it takes much more time to use the algorithm to compute 502 – 304. A more efficient way is to reason that 500 – 300 = 200 and that 200 – 2 = 198. An understanding of place value and reference points such as 25 × 4 = 100 and 250 × 4 = 1000 allow children the flexibility to determine for themselves the most efficient method for solving a problem in a given situation.

The second argument for teaching algorithms was to give struggling students a method for getting answers. It seemed that these students deserved to be given a method for at least getting an answer to achieve some degree of success. It later became evident, however, that when these students forgot a step or developed a "buggy" algorithm, they had nothing to fall back on. Teaching algorithms to these students also sent them the message that "the logic of this procedure is too much for you; so just follow these steps and you'll get the right answer." Some students need more time than others to develop the logic of mathematics. These children deserve the time they need to develop confidence in their ability to make sense of mathematics.

CONCLUSION

Children come to school with enormous potential for powerful thinking. Educators must try to develop this potential instead of continuing to "put the cart in front of the horse." Adults may pack the cart with treasures, but children need to go through their own constructive process and to proceed with confidence in their own ability to solve problems every step of the way.

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